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A NEW CANONICAL FORM OF THE ELLIPTIC INTEGRAL

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The elliptic norm curve E_n in space S_{n-1} admits a group G_{2n^2} of collineations, and in fact there is a single infinity of such curves which admit the same group. A particular E_n of the family is distinguished by a value of the parameter τ, itself an elliptic modular function defined by the modular group congruent to identity (mod n).

In the group G_{2n^2} there are certain involutory collineations with two fixed spaces. If E_n is projected from one fixed space upon the other, a family of rational curves R_m mapping the family of E_n's, is obtained. The quadratic irrationality separating involutory points on E_n involves the modulus τ and the parameter t of the R_m. When the genus of the modular group is zero and $n = 3, 4, 5$, the irrationality can be used to define the elliptic parameter

$$u_1 = \int \frac{(t\,dt)}{(t\,\tau)\,\alpha_\tau^{r-3}\,\alpha_t^3},$$

where α_t^r is the tetrahedral, octahedral, or icosahedral form. This is in contrast to Klein's form[1] as developed by Bianchi,[2] for there the normal elliptic integral is a rational curvilinear integral along an elliptic curve.

A comparison of the two integrals is more illuminating if it is carried out for a special case. Let E_n be E_5 in S_4. In Bianchi's notation the five quadrics having E_5 as their common intersection are

$$\varphi_i : a\,x_i^2 + a^2\,x_{i+2}\,x_{i+3} - x_{i+1}\,x_{i+4} = 0,\ (x_{i+5} \equiv x_i),\ (i = 0, \cdots 4),$$

where a is the modulus. If a transformation of coordinates is made in order to bring into evidence the fixed spaces of the involutory collineation used in the projection, then the icosahedral form which appears in the irrationality is

$$\alpha_t^{12} = t_1\,t_2\,(t_1^{10} + 11\,t_1^5\,t_2^5 - t_2^{10}).$$

The integral u_1 involving $\tau = a$ explicitly in a rather simple form is uniquely defined. Moreover it is invariant under all cogredient trans-

formations of t and τ, which leave the form α_x^{12} unaltered, i.e., the sixty transformations of the icosahedral group applied simultaneously to t and τ, the parameter of the doubly-covered conic R_2 and the modulus of the elliptic quintic curve E_5, leave u_1 unaltered.

Consider now Bianchi's integral. It is defined as

$$U = C \int \frac{(u\, dv - v\, du)}{(\varphi_0\, \varphi_1\, \varphi_2\, u\, v)},$$

where C is a constant, u and v any two expressions linear in x, and the denominator is the functional determinant of φ_0, φ_1, φ_2, u and v. For a particular choice of u and v the integral assumes the simple form

$$U = C \int \frac{(x_0\, dx_1 - x_1\, dx_0)}{5\, a^3\, x_2\, x_4 - (2\, a^5 + 1)\, x_0\, x_1},$$

where the x's are subject to the relations φ_i. Different expressions for U can be obtained by making different choices for u and v. Hence there is no unique form for U as there is for u_1. The integral U assumes various conjugate forms under the Group G_{50} of collineations on the x's, and also under the transformations of a.

So the integral u_1 seems to have an advantage over U in its simplicity of form, its uniqueness, and its invariancy under transformations.

By a study of the integral u_1 itself some interesting results are derived. The modular equation connecting τ and J, the absolute invariant of u_1, can be deduced as the result of the binary syzygy of lowest weight connecting the concomitants of α_x^r. The requirement that the Riemann surface attached to the modular equation be regular leads to the modular equations associated with the regular bodies. It is then possible to eliminate the more tedious individual proofs used by Bianchi in the discussion of the moduli of E_3 and E_5 to show that these moduli are the tetrahedral and icosahedral irrationalities respectively. In fact the algebraic discussion carried out once for α_x^r is complete for factor groups of genus zero, which have been discussed by Klein,[3] i.e., those isomorphic with the groups associated with the regular bodies, namely, the one dihedral group G_6 and the tetrahedral, octahedral, and icosahedral groups.

[1] Klein-Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunctionen*, Bd. 2, Abschnitt 5.

[2] Bianchi, Über die Normalformen dritter und fünfter Stufe des elliptischen Integrals erster Gattung, *Math. Ann., Leipzig*, **17**, 234–262, (1880).

[3] Klein-Fricke, loc. cit., vol. 1, pp. 339 ff.